数理解析研究所講究録
第 844 巻 1993 年 206-217

Horizontal divisors on arithmetic surfaces associated with Belyi uniformizations*)

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For a finite surjective morphism $f: Y \to X$ between some *arithmetic* surfaces and a *horizontal* prime divisor D on X, we consider questions related to connectedness of $f^{-1}(D)$. The results will then be applied to fundamental groups of related surfaces. This article owes much to Harbater's work [Hb], and contains an appendix on some proof by T. Saito.

By an arithmetic surface, we mean any two dimensional integral scheme of finite type having structure of a flat $\mathfrak{O}$-scheme, where $\mathfrak{O}$ is the ring of integers of a number field k (the dimension relative to $\mathfrak{O}$ is 1). Horizontal divisors are those finite over $\mathfrak{O}$. Let us begin by describing some special examples. First, if $\mathbf{P}^1_{\mathbf{Z}}$ is the projective line over $\mathbf{Z}, f: \mathbf{P}^1_{\mathbf{Z}} \to \mathbf{P}^1_{\mathbf{Z}}$ is defined by $y \to y^N = x$ $(N \geq 1)$, and D is defined by $x = 1$, then $f^{-1}(D) \simeq \mathrm{Spec}\ (\mathbf{Z}[y]/(y^N - 1))$ is connected, being the spectrum of the ring of virtual characters of a finite group ($\simeq \mathbf{Z}/N$ in this case; cf [S] 11·4). Each irreducible component of $f^{-1}(D)$ meets some other components on the special fibers $\mathbf{P}^1_{\mathbf{Z}} \otimes \mathbf{F}_p$ at $p|N$, to make $f^{-1}(D)$ connected. This remains valid if $\mathbf{Z}$ is replaced by any $\mathfrak{O}$. Secondly, if $f: Y \to X$ is *everywhere etale* and D is normal, then distinct irreducible components of $f^{-1}(D)$ cannot meet each other (cf. e.g. [G] Cor 9·11). As these examples show, when $f^{-1}(D)$ splits into the union of several irreducible components, the connectedness of $f^{-1}(D)$ is closely related to ramifications of f at special fibers (vertical prime divisors) of Y. In a sense, it gives a "horizontally patched" information on such ramifications.

*) Interium report

The main results proved in this note are as follows. Let $X = \mathbf{P}^1_{\mathfrak{O}}$ be the projective t-line over $\mathfrak{O}$ ($\mathfrak{O}, k$ being as above), $L/k(t)$ be a finite extension *unramified outside* $t = 0, 1, \infty$ (the "Belyi uniformization"), and $f : Y \to X$ be the integral closure of X in L. For $a \in k^{\cup}(\infty)$, denote by D_a the prime divisor on X defined by $t = a$. Then

Theorem A (Th 2, Prop 1 of §2). *(i) If* $a = 0, 1, \infty, f^{-1}(D_a)$ *is connected; (ii) if* $a \in \mathbf{Q}, f^{-1}(D_a)$ *is again connected; (iii) there exists* $\mathfrak{O}$ *and* $a \in \mathfrak{O}$, *such that* $a, 1 - a$ *are both units of* $\mathfrak{O}$ *(so that* D_a *does not meet* $D_0^{\cup} D_1^{\cup} D_\infty$*), and that* $f^{-1}(D_a)$ *is connected for any* f.

As direct applications, we obtain, for example:

Theorem B (i) (T. Saito). $\pi_1(\mathbf{P}^1_{\mathfrak{O}} - D_0^{\cup} D_1^{\cup} D_\infty) \simeq \pi_1(\mathrm{Spec}\,\mathfrak{O})$; *(ii) if one of* $t = 0, 1, \infty$ *is totally ramified in* $L/k(t)$, *then* $\pi_1(Y) \simeq \pi_1(\mathrm{Spec}\,\mathfrak{O})$.

See §3 for more details (Proposition 2, Cor 1,2,3). Saito's original proof of (i) is quite different (see §3, and Appendix).

As for (ii), according to Belyi [B](Th 4 and its proof), every algebraic function field of one variable L over a number field k contains such an element t that $L/k(t)$ is unramified outside $t = 0, 1, \infty$ and, in fact, moreover, totally ramified at $t = \infty$ when L has a prime divisor of degree 1 over k. So, (ii) implies that every arithmetic surface over $\mathfrak{O}$ (having a section over $\mathfrak{O}$) has a normal model Y such that $\pi_1(Y) \simeq \pi_1(\mathrm{Spec}\,\mathfrak{O})$.

In §1, we shall prove a criterion for connectedness of $f^{-1}(D)$ when $X = \mathbf{P}^1_{\mathfrak{O}}$ (Theorem 1). This is just a direct consequence of Harbater's criterion [Hb] for an algebraic function given as power series over $\mathfrak{O}$ to be rational (a modification of Dwork's criterion). Logically, this is just a simple remark. But the author could not find a reference with explicit statement on this connection, and so he thought it necessary to be presented. We note here that in the *geometric* cases (geometric surfaces, etc.), the connectedness of $f^{-1}(D)$ was established under some mild conditions (such as $(D^2) > 0$) in Hironaka-Matsumura

[H-M] cf. also [Ht]. There, the main point was the extendability of any formal-rational function on the completion of X along D to a global rational function on X. In our arithmetic case, one must also take care of neighborhoods of D above *archimedean* places of $\mathfrak{O}$ which is the role of archimedean radii of convergence appearing in the criterion.

In §2, we restrict ourselves to the case where only $t = 0, 1, \infty$ can be ramified in $f \otimes k$ ("Belyi uniformization"), and obtain Theorem 2, Proposition 1.

In §3, we prove Proposition 2 and its corollaries as direct applications of §2.

The next problem would be to find out whether Theorem 1 extends to more general arithmetic surfaces and a full arithmetic analogue of Hironaka-Matsumura criterion can be described using an appropriate Arakelov type theory. We hope to be able to discuss this problem more concretely in the near future.

The author wishes to thank Kyoji Saito and Takeshi Saito for helpful discussions.

§1. In what follows, k will denote an algebraic number field, $\mathfrak{O}$ the ring of integers of k, and Σ the set of all distinct embeddings $\sigma : k \hookrightarrow \mathbf{C}$. We denote by $K = k(t)$ the rational function field of one variable, and by L/K a finite extension which may contain constant field extensions. Let $X = \mathbf{P}^1_{\mathfrak{O}} = \operatorname{Spec}\mathfrak{O}[t] \cup \operatorname{Spec}\mathfrak{O}[t^{-1}]$, and $f : Y \to X$ be the integral closure of X in L. For each $\sigma \in \Sigma$, let $f_\sigma : Y_\sigma \to X_\sigma$ denote the base change $\otimes_{k,\sigma}\mathbf{C}$ of f. Each f_σ defines a finite branched covering $Y_\sigma(\mathbf{C}) \to X_\sigma(\mathbf{C}) = \mathbf{P}^1(\mathbf{C})$ between (not necessarily connected) compact Riemann surfaces. For $r > 0$, put $B(r) = \{z \in \mathbf{C}; |\, z \,| < r\} \subset \mathbf{P}^1(\mathbf{C})$.

Theorem 1. *Let D_0 be the prime divisor of $X = \mathbf{P}^1_{\mathfrak{O}}$ defined by the equation $t = 0$. Assume that there exists $r_\sigma > 0$ for each $\sigma \in \Sigma$ such that f_σ is unramified above $B(r_\sigma)$ and $\Pi_\sigma r_\sigma \geq 1$. Then the $\mathfrak{O}$-scheme $f^{-1}(D_0) = Y \times_X D_0$ is connected.*

(Note that if L/K is a constant field extension, then $f^{-1}(D_0)$ is the spectrum of the corresponding ring of integers.)

This theorem is a direct consequence of the following result of Harbater ([Hb] Prop 2.1 and the preceding remarks).

Lemma (Harbater). *Let k be a number field with normalized absolute values $|\ |_v$ (so that $\Pi_v \mid a \mid_v = 1$ for all $a \in k^\times$). Suppose that $F(t) \in k[[t]]$ is algebraic over $k(t)$. Then one can choose $r_v > 0$ for each place v of k, with $r_v = 1$ for almost all v, such that $F(t)$ is v-adically convergent on the open disc of radius r_v (w.r.t. $|\ |_v$). If, moreover, one can choose r_v's such that $\Pi_v r_v \geq 1$, then $F(t)$ is rational, i.e. $F(t) \in k(t)$.*

Remark 1. For a complex archimedean place v corresponding to $\sigma, \bar{\sigma} \in \Sigma, r_v$ in this lemma corresponds to $r_\sigma r_{\bar{\sigma}} = r_\sigma^2$ in Theorem 1.

Remark 2. We shall only need the case where $F(t)$ belongs to $\mathfrak{O}[[t]]$ and is integral over $\mathfrak{O}[t]$. In this case, since we may choose $r_v = 1$ for all non-archimedean v, the assumption is $\Pi_\sigma r_\sigma \geq 1$. (It is not easy to make use of non-archimedean v with $r_v > 1$; see Remark 4 at the end of §1.) In this case, the proof in [Hb] is easy enough to be sketched. For each $\sigma, F_\sigma \in \mathbf{C}[[t]]$ is not only holomorphic in the open disc of radius r_σ, but extends to a continuous function on its closure, because F_σ is integral over $\mathbf{C}[t]$. Therefore, by the Riemann-Lebesgue lemma, one obtains $\mid a_n^\sigma \mid r_\sigma^n \to 0 (n \to \infty)$. Therefore, $\Pi_\sigma \mid a_n^\sigma \mid = N(a_n) \to 0$. But since $a_n \in \mathfrak{O}$, and hence $N(a_n) \in \mathbf{Z}$, this implies $N(a_n) = 0$ for $n \gg 0$, hence $F(t) \in \mathfrak{O}[t]$. For more details, and for comparison with classical Dwork criterion, see [Hb] §2.

Proof of Theorem 1. Choose any geometric point η of $Y_k = Y \otimes_{\mathfrak{O}} k$ above $t = 0$, and use the completion of L at η to embed L into $\bar{k}((t))$ ($\bar{k}$: an algebraic closure of k).

Claim 1A. $L \cap \mathfrak{O}[[t]] \subset k(t)$.

Proof. Take any $F = F(t) = \Sigma a_n t^n \in L \cap \mathfrak{O}[[t]]$, and by multiplying a suitable element $\neq 0$ of $\mathfrak{O}[t]$, we assume F to be integral over $\mathfrak{O}[t]$. Let $\sigma \in \Sigma$. Then $F_\sigma(t) = \Sigma a_n^\sigma t^n \in \mathbf{C}[[t]]$ extends to a holomorphic function on $B(r_\sigma)$ (and hence converges on $B(r_\sigma)$), because F_σ is integral over $\mathbf{C}[t]$ and f_σ is unramified above $B(r_\sigma)$. Since $\Pi_\sigma r_\sigma \geq 1$, the above lemma gives $F(t) \in k[t]$.

Claim 1B. *Let E be the quotient field of $\mathfrak{O}[[t]]$ ($k(t) \subset E \subset k((t))$). Then $L \cap E = k(t)$.*

Proof. Since $L \cap E$ is finite over $k(t)$, every element of $L \cap E$ is a $k(t)^\times$- multiple of some $g \in L \cap E$ which is integral over $\mathfrak{O}[t]$. Since $g \in E$ and integral over $\mathfrak{O}[[t]]$, $g \in \mathfrak{O}[[t]]$. Hence $g \in L \cap \mathfrak{O}[[t]] \subset k(t)$ by Claim 1A.

Claim 1C. *L and E are linearly disjoint over $k(t)$.*

Proof. Apply Claim 1B to the Galois closure of L over $k(t)$ (which does not change r_σ's).

Claim 1D. *Let B be the integrel closure of $\mathfrak{O}[t]$ in L. Then $B \otimes_{\mathfrak{O}[t]} \mathfrak{O}[[t]] \simeq \varprojlim(B/t^N B)$ is an integral domain.*

Proof. Since $B \to L$ is injective and $\mathfrak{O}[[t]]/\mathfrak{O}[t]$ is flat, $B \otimes_{\mathfrak{O}[t]} \mathfrak{O}[[t]] \to L \otimes_{\mathfrak{O}[t]} \mathfrak{O}[[t]]$ is also injective. On the other hand, $\mathfrak{O}[[t]] \to E$ is injective and $L/\mathfrak{O}[t]$ is flat; hence $L \otimes_{\mathfrak{O}[t]} \mathfrak{O}[[t]] \to L \otimes_{\mathfrak{O}[t]} E = L \otimes_{k(t)} E$ is also injective. By Claim 1C, $L \otimes_{k(t)} E$ is a field. Therefore, $B \otimes_{\mathfrak{O}[t]} \mathfrak{O}[[t]]$ is a domain.

The last isomorphism follows from a general fact; if A is a noetherian ring, M is a (not necessarily free) finite A-module, and I is an ideal of A, then $M \otimes \varprojlim(A/I^n) \simeq \varprojlim(M/I^n M)$ (cf [A-M] p108).

Claim 2. *If J, J' are ideals of B such that (i) $J + J' = (1)$, (ii) $J, J' \supset (t)$, (iii) $(JJ')^n \subset (t)$ for some $n \geq 1$, then either $J = (1)$ or $J' = (1)$.*

Proof. By these conditions,

$$\varprojlim(B/t^N B) \simeq \varprojlim(B/J^N) \oplus \varprojlim(B/J'^N)$$

which reduces the Claim to Claim 1D.

Completing the proof of Theorem 1. If $f^{-1}(D_0) = \mathrm{Spec}(B/tB)$ were not connected, it must be a disjoint union of two non-empty subsets S, S'. Let J (resp. J') be the intersection of all (minimal) primes of B belonging to S (resp. S'). Then J, J' satisfies the conditions of Claim 2. Therefore, J or $J' = (1)$, a contradiction. $\square$

Remark 3. Perhaps we should show some example where $f^{-1}(D)$ is disconnected. This is the case when $L = \mathbb{Q}(t, y)$, with $y^2 - y = t$ and D is defined by $t = 0$. In fact, then $f^{-1}(D) \simeq \mathrm{Spec}(\mathbb{Z}[y]/y(y-1)) \cong \mathrm{Spec}\,\mathbb{Z} \sqcup \mathrm{Spec}\,\mathbb{Z}$. Note that the branch point $t = -\frac{1}{4}$ is "archimedean close" to $t = 0$.

Remark 4. At non-archimedean primes $\mathfrak{p}$, the radius of convergence can be strictly smaller than the distance from the center of the nearest branch point (cf. [Hb] §3 Remark 2, [D-R]). For this reason, we could not use non-archimedean primes to loosen the assumption of Theorem 1.

§2. Let $k, \mathfrak{O}$, L/K, $f : Y \to X$ ($X = \mathbb{P}^1_{\mathfrak{O}}$) be as at the beginning of §1, and now we assume that $f_k; Y_k \to X_k$ is unramified outside $t = 0, 1, \infty$. A prime divisor of X defined by $t = 0, 1$, or ∞ will be called *cuspidal.*

Theorem 2. *If f_k is unramified outside $t = 0, 1, \infty$, and D is a cuspidal prime divisor of $X = \mathbb{P}^1_{\mathfrak{O}}$, then $f^{-1}(D)$ is connected.*

Proof. We may assume that D is the cusp defined by $t = 0$. Replacing t by $t^{1/N}$ with a suitable N, we are reduced to the situation where f_k is unramified outside $t \in \mu_N$ (the group of N-th roots of unity). But then the connectedness of $f^{-1}(D)$ is an immediate consequence of Theorem 1. $\square$

For the closure D_a in $\mathbf{P}^1_{\mathfrak{O}}$ of other rational points $t = a \in k$ $(a \neq 0, 1)$ of $\mathbf{P}^1_k$, we can only prove:

Proposition 1. *If f_k is unramified outside $t = 0, 1, \infty$, and $a \in k$ $(a \neq 0, 1), f^{-1}(D_a)$ is connected at least in the following cases; (i) $a \in \mathbf{Q}$; (ii) $a = (1-\zeta)^{-1}$, where ζ is a root of unity whose order is not a prime power; (ii)′ $a = (1-\zeta')(\zeta-\zeta')^{-1}$, where ζ, ζ' are roots of unity such that none of the orders of $\zeta, \zeta', \zeta'\zeta^{-1}$ are prime powers.*

Remark 5. In cases (ii)(ii)′, a is a *special unit*, i.e., a and $1-a$ are both units. This means that D_a does not meet any cuspidal prime divisor. An example of (ii): $a = (1+\omega)^{-1} = -\omega$, where ω is a cubic root of unity.

By Theorem 1, $f^{-1}(D_a)$ is connected if there exists $\gamma \in GL_2(\mathfrak{O})$ (acting on $\mathbf{P}^1_{\mathfrak{O}}$ by linear fractional transformations) such that $\gamma(a) = 0$ and

$$\prod_{\sigma \in \Sigma} \mathrm{Min}(|\gamma(0)^\sigma|, |\gamma(1)^\sigma|, |\gamma(\infty)^\sigma|) \geq 1.$$

We shall show, in each of the cases (i)(ii)(ii)′, that such an element γ exists.

Actually, we can also show that when a is a special unit, (ii)(ii)′ *are the only cases* where there exists some field $k \ni a$ and some $\gamma \in GL_2(\mathfrak{O})$ satisfying these conditions. Thus, in particular, when a is (a special unit which is) non-abelian over $\mathbf{Q}$, or when (for example) $a = \frac{1}{2}(1+\sqrt{5})$, there does not exist any such γ. We do not know whether $f^{-1}(D_a)$ is connected in such cases.

(i) *The case* $a \in \mathbf{Q}$ $(a \neq 0, 1)$. Write $a = -q/p$ $(p, q \in \mathbf{Z},\ (p,q) = 1,\ q > 0)$. It suffices to find an element $\gamma \in SL_2(\mathbf{Z})$ satisfying $\gamma(a) = 0,\ |\gamma(i)| \geq 1$ $(i = 0, 1, \infty)$. Define $q' \in \mathbf{Z}$

by $0 \leq q' < q$, $pq' \equiv 1 (\mathrm{mod}\, q)$, and $p' \in \mathbf{Z}$ by $p' = (pq'-1)/q$. Then

$$\gamma = \begin{pmatrix} p & q \\ p' & q' \end{pmatrix} \in SL_2(\mathbf{Z}),$$

$\gamma(a) = 0$, and $\gamma(0) = q/q'$, $\gamma(\infty) = p/p'$, $\gamma(1) = (p+q)/(p'+q')$. But $|q'/q| < 1$ and $|p'/p| = |q'/q - 1/pq| \leq 1$; hence $|\gamma(0)|, |\gamma(\infty)| \geq 1$. Moreover,

$$(p'+q')/(p+q) = q'/q - 1/q(p+q);$$

hence

$$-1 \leq q'/q - 1/q \leq (p'+q')/(p+q) \leq q'/q + 1/q \leq 1;$$

hence $|\gamma(1)| \geq 1$. Therefore, γ satisfies the desired properties.

(ii) In this case, it is enough to take $\gamma(t) = 1 - a^{-1}t$. In fact, then $\gamma(a) = 0$, $\gamma(0) = 1$, $\gamma(1) = \zeta$, $\gamma(\infty) = \infty$.

(ii)′ In this case, it is enough to take

$$\gamma = \begin{pmatrix} \zeta - \zeta' & \zeta' - 1 \\ \zeta - \zeta' & \zeta(\zeta' - 1) \end{pmatrix}.$$

In fact, then $\det \gamma = (\zeta - 1)(\zeta' - 1)(\zeta - \zeta') \in \mathfrak{O}^\times$, $\gamma(a) = 0$, $\gamma(0) = \zeta^{-1}$, $\gamma(1) = \zeta'^{-1}$, $\gamma(\infty) = 1$. □

§3. In general, let Y, Z be connected locally noetherian schemes, $f : Z \to Y$ be a morphism and $f_* : \pi_1(Z, \zeta) \to \pi_1(Y, \eta)$ be the induced homomorphism between their fundamental groups, where ζ is any geometric point of Z and $\eta = f(\zeta)$. Then by their definitions [G], f_* is *surjective* if and only if $Z' = Z \times_Y Y'$ is *connected* for any finite etale connected covering Y'/Y of Y. We apply this to the determination of $\pi_1(Y)$ for some special arithmetic surfaces Y, by using horizontal prime divisors $Z \hookrightarrow Y$ and the results of §2.

The following is a direct application.

Proposition 2. *Let k be a number field, $\mathfrak{O}$ its ring of integers, and $X = \mathbf{P}^1_{\mathfrak{O}}$ (the projective t-line over $\mathfrak{O}$). Let $L/k(t)$ be a finite extension field, which is unramified outside $t = 0, 1, \infty$, and $f : Y \to X$ be the normalization of X in L. Let $a \in k^{\cup}(\infty)$ be either $a \in \mathbf{Q}^{\cup}(\infty)$ (including $0, 1, \infty$) or of the form* (ii) *or* (ii)$'$ *of Proposition* 1, *and D_a be the prime divisor on X defined by $t = a$. Let E be any closed subscheme of Y contained in (the support of) $f^{-1}(D_0 {}^{\cup} D_1 {}^{\cup} D_\infty)$, which does not meet $f^{-1}(D_a)$ (for example, $E = \emptyset$). Then the natural homomorphism*

$$\pi_1(f^{-1}(D_a)^{\mathrm{red}}) \longrightarrow \pi_1(Y - E)$$

is surjective. In particular, (i) *if $f^{-1}(D_a)^{\mathrm{red}} \xrightarrow{\sim} \operatorname{Spec}\mathfrak{O}$, then $\pi_1(Y - E) \xrightarrow{\sim} \pi_1(\operatorname{Spec}\mathfrak{O})$;* (ii) *if $f^{-1}(D_a)^{\mathrm{red}}$ is a tree-like union of* $\operatorname{Spec}\mathfrak{O}$ *(see below) and $\pi_1(\operatorname{Spec}\mathfrak{O}) = (1)$, then $\pi_1(Y - E) = (1)$.*

Here, $f^{-1}(D_a)^{\mathrm{red}}$ (the reduced part of $f^{-1}(D_a)$) is called *tree-like* if its graph (edges = irreducible components, vertices on an edge = closed points on the corresponding irreducible component) is a tree.

Proof. The prime divisor $F = f^{-1}(D_a)^{\mathrm{red}}$ is a closed subscheme of $Y_1 = Y - E$. If Y_1'/Y_1 is any connected finite etale covering, $Y_1' \times_{Y_1} F \simeq Y' \times_Y F$, where Y' is the integral closure of Y (and also of $\mathbf{P}^1_{\mathfrak{O}}$) in the function field of Y_1'. By Proposition 1, $Y' \times_Y f^{-1}(D_a) = Y' \times_X D_a$ is connected; hence $Y' \times_Y F$ is also connected. Therefore, $\pi_1(F) \to \pi_1(Y_1)$ is surjective.

When $F \xrightarrow{\sim} \operatorname{Spec}\mathfrak{O}$, this defines a section $\operatorname{Spec}\mathfrak{O} \to Y_1$, and hence we have a surjection $\alpha : \pi_1(\operatorname{Spec}\mathfrak{O}) \to \pi_1(Y_1)$, and the structural homomorphism $\beta : \pi_1(Y_1) \to \pi_1(\operatorname{Spec}\mathfrak{O})$, with $\beta \circ \alpha = id$. Therefore, $\pi_1(Y_1) \xrightarrow{\sim} \pi_1(\operatorname{Spec}\mathfrak{O})$. In case (ii), F has no non-trivial connected finite etale coverings, because each irreducible component $\simeq \operatorname{Spec}\mathfrak{O}$ is simply connected, and there can be no non-trivial connected "mock coverings" (graph-theoretically produced finite connected etale coverings) because F is tree-like. $\square$

Corollary 1 (T. Saito). $\pi_1(\mathbf{P}^1_{\mathfrak{O}} - D_0 \cup D_1 \cup D_\infty) \simeq \pi_1(\operatorname{Spec} \mathfrak{O})$.

This fact may well have been known, but the author could not find any reference, except that Example 3.1 in [Hb] §3 is quite close. (It gives $\pi_1(\operatorname{Spec} \mathbf{Z}[t, (t^N - 1)^{-1}]) = (1)$, to which the case $\mathfrak{O} = \mathbf{Z}$ reduces directly, and [Hb] contains enough tools for treating the case of general $\mathfrak{O}$.) As far as the author knows, the first proof of this was provided by T. Saito. It is a direct application of generalized Abhyankar lemma (see Appendix). Our argument gives it an alternative proof which is more archimedean in nature.

Proof. First, take some a as in Prop. 1 (ii) or (ii)$'$, and choose k such that $k \ni a$. In Prop. 2, take $Y = X$, $E = D_0 \cup D_1 \cup D_\infty$. Since $D_a \cap E = \emptyset$, Prop. 2 (i) applies to this case, and we conclude that $\pi_1(\mathbf{P}^1_{\mathfrak{O}} - E) \simeq \pi_1(\operatorname{Spec} \mathfrak{O})$ for $\mathfrak{O}$: big enough. But then, for any $\mathfrak{O}$, $\mathbf{P}^1_{\mathfrak{O}} - E$ cannot have finite etale connected coverings other than constant ring extensions (which must be etale). Therefore, our assertion holds for any $\mathfrak{O}$. □

Corollary 2. *Let $f : Y \to X$ be as at the beginning of Prop. 2 (the first two sentences preserved). Suppose that one of the cusps, say $t = \infty$, is totally ramified in $f_k = f \otimes k$: $Y_k \to X_k$. Then $\pi_1(Y) \xrightarrow{\sim} \pi_1(\operatorname{Spec} \mathfrak{O})$, or more strongly,*

$$\pi_1(Y - D_0 \cup D_1) \cong \pi_1(\operatorname{Spec} \mathfrak{O}).$$

Proof. In fact, in this case $f^{-1}(D_\infty)^{\mathrm{red}} \simeq \operatorname{Spec} \mathfrak{O}$.

In particular,

Corollary 3. *Let p be a prime, $a, b, c \in \mathbf{Z}$, $a + b + c = 0$, $abc \not\equiv 0 \pmod p$, and $L = \mathbf{Q}(t, y)$, where*

$$y^p = (-1)^c t^a (1 - t)^b$$

(a "primitive Fermat curve"). Let $f : Y \to \mathbf{P}^1_{\mathbf{Z}}$ be the normalization of $\mathbf{P}^1_{\mathbf{Z}}$ (the t-line) in

L. Then for $i, j \in \{0, 1, \infty\}$, $i \neq j$,

$$\pi_1(Y - f^{-1}(D_i \cup D_j)) = (1).$$

[Appendix] *T. Saito's original proof of Cor.* 1 *of Prop.* 2

It proceeds as follows. Let $L/k(t)$, $f : Y \to X = \mathbf{P}^1_{\mathfrak{O}}$ be as at the beginning of Proposition 2. Suppose that $f : Y \to X$ is etale outside $D_0 \cup D_1 \cup D_\infty$. Let $\mathfrak{p}$ be any prime ideal of $\mathfrak{O}$, and put $X_{\mathfrak{p}} = X \otimes_{\mathfrak{O}} (\mathfrak{O}/\mathfrak{p})$. Choose any cuspidal prime divisor D_i $(i = 0, 1, \infty)$ on X, and let P be the intersection of D_i with $X_{\mathfrak{p}}$, which is a closed point on $X_{\mathfrak{p}}$. Then the only prime divisor on X passing through P, along which f is possibly ramified, is D_i. From this follows, by the generalized Abhyankar lemma ([G] Exp. XIII §5), that the ramification indices of $f_k = f \otimes k$ above $t = i$ cannot be divisible by the residue characteristic of $\mathfrak{p}$. Since $\mathfrak{p}$ and i are arbitrary, f must be etale also above D_0, D_1, D_∞; hence $\pi_1(X - D_0 \cup D_1 \cup D_\infty) \simeq \pi_1(X) \simeq \pi_1(\operatorname{Spec} \mathfrak{O})$, as desired.

Saito has also noted that the same argument holds for a somewhat more general case; $\mathbf{P}^1_{\mathfrak{O}} - \bigcup_{a \in A} D_a$ where A is a finite set of elements of $k^{\cup}(\infty)$ satisfying the following conditions. For each pair of $\mathfrak{p}$ and $a \in A$, put $P(a, \mathfrak{p}) = D_a \cap X_{\mathfrak{p}}$ (a closed point on $X_{\mathfrak{p}}$). Then for each pair $(a, \mathfrak{p})$, either $P(a, \mathfrak{p}) \neq P(a', \mathfrak{p})$ for all $a' \neq a$ $(a' \in A)$, or there exists exactly one $a' \in A$, $a' \neq a$ with $P(a', \mathfrak{p}) = P(a, \mathfrak{p})$, *and* in this case the maximal ideal of the local ring of X at $P(a, \mathfrak{p})$ is generated by two elements defining D_a and $D_{a'}$ at $P(a, \mathfrak{p})$. (Roughly speaking, the conditions require that the only singularities of $\bigcup D_a$ are "ordinary double points".)

An example: $\mathfrak{O} = \mathbf{Z}$, $A = \{0, 1, 2, 3, \infty\}$.

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